3. Concluding Remarks. We can obtain formulas similar to those given here for any region (and weight function) which has the same symmetries as the ellipse. We need only substitute the appropriate monomial integrals $I_{2n,2m}$ in the expressions given.

It should also be noted that the formulas of degree 7 are not unique. Similar formulas can be obtained by choosing different values for the quantities k_1 and k_2. Various 12-point formulas are obtained by choosing k_1 and k_2 to satisfy

$$k_1 + k_2 = I_{00} - 4A_5 .$$

Although there is this free parameter in the 12-point formulas we believe it is not possible to obtain a formula of degree 7 using fewer points.

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Improved Asymptotic Expansion for the Exponential Integral with Positive Argument

By Donald van Zelm Wadsworth

The usual asymptotic approximation to the exponential integral can be markedly improved, for the case with positive real argument, by adding a simple correction term as shown below. Similar results for the error function with imaginary argument (essentially the same as Dawson's function) are given in [1].*

By definition, the exponential integral with positive real argument is

$$\text{Ei}(x) = -\int_{-x}^{\infty} t^{-1}e^{-t}\,dt = -\int_L t^{-1}e^{-t}\,dt - i\pi.$$

The line integral along the real axis from $-x$ to ∞ is a Cauchy principal value since there is a pole at the origin. The path of integration L goes from $-x$ to ∞, passing above the origin. Repeated partial integration of the infinite integral yields $\text{Ei}(x) = E_n(x) + e_n(x)$, where

$$E_n(x) = x^{-1}e^x \sum_0^{n-1} m!x^{-m}$$

is the asymptotic approximation for the interval $(n - \frac{1}{2}) \leqq x < (n + \frac{1}{2})$, and

$$e_n(x) = -(-)^n n! \int_L t^{-n-1}e^{-t}\,dt - i\pi$$

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* The correction term derived in [1] could also be obtained, in a less direct fashion, from the Chebyshev polynomial expansions for Dawson's function given in [2].

is the error of the asymptotic approximation. By an analysis similar to that given in [1], we obtain

$$(1)\qquad e_n(x) = (-)^n n! x^{-n} \int_0^{\pi} e^{-x\cos\varphi} \sin(x \sin\varphi + n\varphi)\, d\varphi - (-)^n n! \int_x^{\infty} t^{-n-1} e^{-t}\, dt.$$

By expanding the right side of (1) as in [1], we obtain

$$(2)\qquad e_n(x) = -n! e^x x^{-n-1}(\tfrac{1}{3} + n - x + O(nx^{-1} - 1)).$$

By use of Stirling's approximation for a factorial, we can put (2) in the form $e_n(x) = e_n^*(x) + O(x^{-3/2})$, where

$$e_n^*(x) = -(2\pi/x)^{1/2}(\tfrac{1}{3} + n - x).$$

We take $E_n(x) + e_n^*(x)$ to be the improved asymptotic expansion for the exponential integral with positive real argument.

In the table, $\mathrm{Ei}(x)$ as tabulated in Jahnke and Emde's *Tables of Functions* is compared with $E_n(x)$, $E_{n-1}(x)$, and $E_n(x) + e_n^*(x)$. Even at $x = 1$, the improved approximation has only about one per cent error compared to forty per cent for $E_n(x)$.

Accuracy of Asymptotic Approximations

x	$\mathrm{Ei}(x)$	$E_n(x) + e_n^*(x)$	$E_n(x)$	$E_{n-1}(x)$
1.0	1.895	1.883	2.718	—
1.2	2.442	2.462	2.767	—
1.4	3.007	3.038	2.897	—
1.6	3.605	3.577	5.030	3.096
1.8	4.250	4.232	5.228	3.361
2.0	4.954	4.951	5.542	3.695
2.5	7.074	7.060	8.382	6.822
3.0	9.934	9.932	10.415	8.927
3.5	13.925	13.917	15.033	13.710
4.0	19.631	19.630	20.048	18.768

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